Combinatorial Inequalities of Kazhdan-Lusztig polynomials in Bruhat graphs

Masato Kobayashi
Saitama University

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Key Words
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New idea

- Strict edges
1 Introduction
2 Bruhat graphs
3 KL polynomials
4 Key lemma
5 Theorem
6 Future work
Motivation
Understand behavior of $P_{uw}(1)$ for $u \in X(w)$ in terms of Bruhat graph.

Theorem (Kobayashi)
$\exists$ a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

Idea
When a strict inequality occurs?
When $P_{uw}(1) > P_{vw}(1)$ for $u < v$?

Notation $X(w) = [e, w]$
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in terms of Bruhat graph.

Theorem (Kobayashi)
\[ \exists \text{ a lower bound of } P_{uv}(1) \text{ by graph-theoretic distance.} \]

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When a **strict** inequality occurs?
Notation $X(w) = [e, w]$ 

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**Theorem (Kobayashi)**

$\exists$ a lower bound of $P_{uw}(1)$ by graph-theoretic distance.

**Idea**

When a strict inequality occurs? When $P_{uw}(1) > P_{vw}(1)$ for $u < v$?
$u \rightarrow w$ means $w = ut$ for some $t \in T$ and $\ell(u) < \ell(w)$. 
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**Def (Dyer 91)**

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  - vertices $w \in W$,
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- **Bruhat subgraph for** \([u, w]\)
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**Def (Dyer 91)**

- **Bruhat graph**: vertices $w \in W$, edges $u \to w$.
- **Bruhat subgraph for** $[u, w]$
- **Bruhat path**

\[ u \to v_1 \to \cdots \to v_n = w \]
Figure: [1324, 3412]
Fact (Kazhdan-Lusztig 79)

\[ \exists \{ P_{uw}(q) \mid u, w \in W \} \subseteq \mathbb{Z}[q] \text{ (KL polynomials) with} \]

1. \[ P_{uw}(q) = 0 \text{ if } u \not\leq w, \]
2. \[ P_{uw}(q) = 1 \text{ if } u = w, \]
3. \[ \deg P_{uw}(q) \leq (\ell(u, w) - 1)/2 \text{ if } u < w, \]
4. \[ q^{\ell(u, w) - 1} \cdot P_{uw}(q) = \sum_{u \leq v \leq w} R_{uv}(q) P_{vw}(q), \]

integer coefficient, but...
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4. if \( u \leq w \), then

\[ q^{\ell(u, w)} P_{uw}(q^{-1}) = \sum_{u \leq v \leq w} R_{uv}(q) P_{vw}(q), \]
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**Fact (Irving 88)**

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$\{P_{uw}(1) \mid u \in X(w)\}$: positive integers

**Def**

$[u, w]$ is

- rationally smooth if $P_{uw}(1) = 1$,
- rationally singular if $P_{uw}(1) > 1$. 

**Prop**

Let $u < v$ in $X(w)$. Then $P_{uw}(q) \geq P_{vw}(q)$ (coefficientwise).

$P_{uw}(1) > P_{vw}(1)$ $\iff P_{uw}(1) > 1$. 

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**Def**

\([u, w]\) is \( \begin{cases} \text{rationally smooth} & \text{if } P_{uw}(1) = 1, \\ \text{rationally singular} & \text{if } P_{uw}(1) > 1. \end{cases} \)

**Fact (Braden-MacPherson 01)**

\( u < v \) in \( X(w) \) \( \implies \) \( P_{uw}(q) \geq P_{vw}(q) \) (coefficientwise).
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**Fact (Irving 88)**

All coefficients of KL polynomials in \( W \) are nonnegative.

\[ \{ P_{uw}(1) \mid u \in X(w) \}: \text{positive integers} \]

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**Fact (Braden-MacPherson 01)**

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Let \( u < v \text{ in } X(w) \). Then

\[ P_{uw}(q) > P_{vw}(q) \iff P_{uw}(1) > P_{vw}(1). \]
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Fix $u \in X(w)$. Suppose $P_{uw}(1) > 1$. 
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Def (Kobayashi 12)

$u \rightarrow v$ in $[u, w]$ is strict if $P_{uw}(1) > P_{vw}(1)$. 
Key lemma (Kobayashi, to appear)

\[ P_{uw}(1) > 1 \]
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Equivalently:
\[ \exists t \in T \text{ such that} \]

\[ P_{uw}(1) > P_{ut,w}(1) > 0. \]
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(Idea of Proof)
1. first order derivative of \( R \)-polynomials
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(Idea of Proof)

1. first order derivative of \( R \)-polynomials
2. Deodhar's inequality
Figure: existence of a strict edge

\[ P_{uw}(1) > P_{ut,w}(1) > 0 \]
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X_{\text{smooth}}(w) = \{ u \in X(w) \mid P_{uw}(1) = 1 \}.
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**Def (distance)**

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\text{dist}(u, X_{\text{smooth}}(w)) = \min\{ d \geq 0 \mid u \rightarrow v_1 \rightarrow \cdots \rightarrow v_d \in X_{\text{smooth}}(w) \}\]
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**Def (distance)**

\[ \text{dist}(u, X_{\text{smooth}}(w)) = \min\{d \geq 0 \mid u \rightarrow v_1 \rightarrow \cdots \rightarrow v_d \in X_{\text{smooth}}(w)\}. \]

In particular, \( \text{dist}(u, X_{\text{smooth}}(w)) = 0 \iff P_{uw}(1) = 1. \)
Let $u \leq w$ and $d = \text{dist}(u, X_{\text{smooth}}(w))$. 

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Key Lemma \( \iff \exists \text{ strict edge } u \rightarrow v_1 \text{ in } X(w) \).
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Let $u \leq w$ and $d = \text{dist}(u, X_{\text{smooth}}(w))$. Then

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Proof

Suppose $P_{uw}(1) > 1$.

Key Lemma $\implies \exists$ strict edge $u \rightarrow v_1$ in $X(w)$.

Repeat: $\exists$ strict edge $v_1 \rightarrow v_2$ in $X(w)$.
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Thus

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$$P_{uw}(1) > P_{v_1w}(1) > \cdots > P_{v_dw}(1).$$

Conclude $P_{uw}(1) \geq d + 1$. 
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g: semisimple Lie algebra
Φ: irreducible simply-laced root system
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Λ⁺: dominant integral weights
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\( g \): semisimple Lie algebra
\( \Phi \): irreducible simply-laced root system
\( W \): Weyl group
\( \Lambda^+ \): dominant integral weights
\( < \): root order

**Fact (Stembridge 98)**

Let \( \lambda, \mu \in \Lambda^+ \).

\[
\lambda \triangleright \mu \implies \lambda - \mu = \alpha (\exists \alpha \text{ one positive root})
\]
g: semisimple Lie algebra
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**Fact (Stembridge 98)**

Let \(\lambda, \mu \in \Lambda^+\).

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\lambda \succ \mu \iff \lambda - \mu = \alpha \quad (\exists \alpha \text{ one positive root})
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Any relation to

\[
P_{uw}(1) > P_{ut,w}(1) > 0?
\]
Conclusion

- Bruhat graphs
- KL polynomials
- rationally singular $\iff P_{uw}(1) > 1$
Conclusion

- Bruhat graphs
- KL polynomials
- rationally singular $\iff P_{uw}(1) > 1$
- Key lemma (= existence of a strict edge)
- Theorem (= a lower bound of $P_{uw}(1)$)
Reference

M. Kobayashi,

*Inequalities on Bruhat graphs, R- and KL polynomials,*

(to appear, J. Comb. Th. Ser. A)
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Thank you.